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Completely rank-nonincreasing linear maps on spaces of operators[☆]

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Submitted by P. Rosenthal

Dedicated to Heydar Radjavi, a great friend and a wonderful mathematician

Abstract

We discuss some results and conjectures concerning rank-nonincreasing, rank-preserving, completely rank-preserving and completely rank-nonincreasing linear maps on spaces of operators. We show that the main conjectures are equivalent to a statement about closures of joint similarity orbits of k -tuples of matrices.

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1. Introduction

In recent years, there have been many papers that study linear preserver problems for operator algebras, and a large number of solutions involve the consideration of finite rank operators. Let \mathcal{A} and \mathcal{B} be two operator algebras and (P) a property of operators such as spectrum, invertibility, an operator equation, a class of operators and so on. If a linear map $\Phi : \mathcal{A} \rightarrow \mathcal{B}$ leaves (P) invariant, we say that Φ is a linear preserver or more exactly, is (P)-preserving. The linear preserver problem asks how to characterize the linear preservers.

Rank-nonincreasing linear maps and rank-preserving linear maps are examples of linear preservers, and these were considered by the second author in [7]. We say Φ is

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rank-nonincreasing (or rank-preserving, respectively) if $\text{rank}(\Phi(A)) \leq \text{rank}(A)$ (or $\text{rank}(\Phi(A)) = \text{rank}(A)$, respectively) for every A in \mathcal{A} , where the rank of operator A is the dimension of its range. Rank-nonincreasing linear maps play important roles in the study of linear preservers. In fact, in the proofs of many known results, the first step involves proving that the linear preservers in question have the property that they map every Rank-one operator to an operator of rank one (e.g., see [7, 9–11]). Rank-nonincreasing and rank-preserving maps on $B(X)$ (X a Banach space) were characterized by the second author in [7].

Rank-preserving and rank-nonincreasing linear maps also occur in the study of approximate equivalence or approximate summands of representations of C^* -algebras [3]. Let π and ρ be two unital representations of a C^* -algebra \mathcal{A} on Hilbert spaces H and K , respectively. We say π and ρ are *approximately (unitary) equivalent* if there is a net $\{U_\lambda\}$ of unitary operators from K to H such that, for every a in \mathcal{A} ,

$$\|U_\lambda^* \pi(a) U_\lambda - \rho(a)\| \rightarrow 0.$$

Approximate equivalence was characterized by a famous theorem of Voiculescu [13], and can be expressed completely in terms of the rank function [3]: π and ρ are approximately equivalent if and only if the natural map Φ from $\pi(\mathcal{A})$ onto $\rho(\mathcal{A})$ defined by $\Phi(\pi(a)) = \rho(a)$ is rank-preserving. Moreover, it is shown in [3] that, when \mathcal{A} is separable, Φ is rank-nonincreasing if and only if there is a representation τ such that $\rho \oplus \tau$ is approximately equivalent to π .

Recently, the first author and Larson [4] introduced the notion of completely rank-nonincreasing linear maps. If \mathcal{S} and \mathcal{T} are linear spaces of operators and $\Phi: \mathcal{S} \rightarrow \mathcal{T}$ is a linear map, we define, for each $n \in \mathbb{N}$, a map $\Phi_n: \mathcal{M}_n(\mathcal{S}) \rightarrow \mathcal{M}_n(\mathcal{T})$ by

$$\Phi_n((s_{ij})) = (\Phi(s_{ij})).$$

We say that Φ is *completely bounded* if $\sup_n \|\Phi_n\| \equiv \|\Phi\|_{\text{cb}} < \infty$, and we say that Φ is *completely positive* if each Φ_n is positive, and we say that Φ is *completely rank-nonincreasing* if each Φ_n is rank-nonincreasing. It was proved in [4] that if H, K are separable Hilbert spaces, \mathcal{A} is a separable C^* -subalgebra of $B(K)$, and $\Phi: \mathcal{A} \rightarrow B(H)$ is a completely bounded linear map then Φ is completely rank-nonincreasing if and only if there is a representation π that is approximately equivalent to the identity representation of \mathcal{A} and operators V and W with $\|V\|\|W\| = \|\Phi\|_{\text{cb}}$ such that, for every $A \in \mathcal{A}$,

$$\Phi(A) = V\pi(A)W.$$

Moreover, if Φ is completely positive, we may choose $V = W^*$.

Let $\vec{T} = (T_1, \dots, T_m)$ and $\vec{S} = (S_1, \dots, S_m)$ be m -tuples of operators with T_i and $S_i \in \mathcal{B}(X)$, the Banach algebra of all bounded linear operators on Banach space X . We say that \vec{T} and \vec{S} are asymptotically joint-similar if there exist nets $\{A_\lambda\}$ and $\{B_\lambda\}$ of invertible operators in $\mathcal{B}(X)$ such that $\lim_\lambda A_\lambda T_j A_\lambda^{-1} = S_j$ and $\lim_\lambda B_\lambda S_j B_\lambda^{-1} = T_j$ ($1 \leq j \leq m$) under a suitable operator topology (often one uses operator norm

topology, strong operator topology (SOT) or weak operator topology (WOT)). Let \mathcal{S} be a linear subspace and $\Phi : \mathcal{S} \rightarrow \mathcal{B}(X)$ a linear map. We say that Φ is a point-weak limit of similarities if there exists a net $\{A_\lambda\}$ of invertible operators in $\mathcal{B}(X)$ such that

$$\Phi(T) = (\text{WOT})\text{-}\lim_{\lambda} A_\lambda T A_\lambda^{-1}$$

for every T in \mathcal{S} . One can define point-strong limit of similarities and point-norm limit of similarities in the same way. Limits of similarities are obviously relative to asymptotic similarities of operators. It turns out that the characterization of limits of similarities reduces to the discussion of rank-nonincreasing and rank-preserving linear maps on subspaces of $\mathcal{F}(X)$ [5].

Another question is the characterization of elementary operators on operator algebras. Let \mathcal{A} be a (unital) Banach algebra. A linear map Φ from \mathcal{A} into itself is called an *elementary operator* if there exists a set $\{a_1, \dots, a_n, b_1, \dots, b_n\}$ of elements in \mathcal{A} such that $\Phi(x) = \sum_{i=1}^n a_i x b_i$ for every $x \in \mathcal{A}$. For an elementary operator Φ , the representation $\Phi(x) = \sum_{i=1}^n a_i x b_i$ may not be unique. The length (or degree) of an elementary operator Φ is the minimum of positive integer n such that there exist a_i and b_i such that $\Phi(x) = \sum_{i=1}^n a_i x b_i$ for every x . The properties of the elementary operators often reveal the properties of the algebras they act on. Elementary operators are continuous under any operator topology and are important ones that are completely bounded. They are natural linear maps on matricial operator algebras and matricial operator spaces. In the C^* -algebra case, elementary operators are closely related to the Haagerup tensor products of C^* -algebras [1]. In spite of their importance, we do not know other equivalent conditions than the original definition for elementary operators. An intensive discussion of rank-nonincreasing linear maps enable us to give a characterization of elementary operator on $\mathcal{B}(X)$ in terms of rank function. We believe that there are similar characterizations for elementary operators on, at least, von Neumann algebras.

This paper is a continuation of [3–5,7,8]. In [7] the weakly continuous rank-nonincreasing linear maps on $\mathcal{B}(X)$ are characterized, where X is a Banach space. In [4], the completely rank-nonincreasing linear maps and the completely k -rank-nonincreasing linear maps on $\mathcal{F}(H)$ are introduced and discussed, where H is a Hilbert space. A characterization of elementary operators on $\mathcal{B}(H)$ is given in [4] which states that a linear map on $\mathcal{B}(H)$ is an elementary operator of length at most k if and only if it is σ -weakly continuous, completely bounded and completely k -rank-nonincreasing.

In the present paper, we first give a general discussion, in Section 2, of the rank-nonincreasing linear maps from $\mathcal{F}(X)$ into $\mathcal{F}(Y)$, where X and Y are real or complex Banach spaces. Some basic representation theorems for such linear maps are obtained. In Section 3, we introduce and discuss the completely rank-nonincreasing linear maps under the framework of Banach spaces. Applying the results in Section 2, we get some sufficient and necessary conditions for a linear map from certain linear subspace $\mathcal{S} \subset \mathcal{F}(X)$ into $\mathcal{F}(Y)$ to be completely rank-nonincreasing. We

also pose some related questions and give a few partial answers. Our results are new even for finite dimensional case. For example, we show that every unital completely rank-nonincreasing linear map on a subspace of matrix algebra can be extended to a completely rank-nonincreasing homomorphism on the algebra generated by the subspace. This result is applied to obtain some results concerning joint-similarity of matrix tuples. The main purpose of Section 4 is to give a characterization of elementary operators. Based on introduction and discussion of the completely k -rank-nonincreasing linear maps, we show that a linear map from $\mathcal{B}(X)$ into $\mathcal{B}(Y)$ is an elementary operator of length at most k if and only if it is σ -weakly continuous and completely k -rank-nonincreasing. This result is much stronger than that for Hilbert spaces in [4] by omitting the “completely bounded” assumption.

2. Rank-nonincreasing linear maps

Let X and Y be Banach spaces over real or complex field \mathbb{F} (i.e., $\mathbb{F} = \mathbb{R}$ or \mathbb{C}), denote by $\mathcal{B}(X, Y)$ ($\mathcal{B}(X)$ when $X = Y$) the Banach space of all bounded linear operators from X into Y . In the case $Y = \mathbb{F}$ we will denote X^* for $\mathcal{B}(X, \mathbb{F})$, the dual space of X . Denote by $\mathcal{F}(X, Y)$ ($\mathcal{F}(X)$ when $X = Y$) the subspace of all (bounded) finite rank linear operators in $\mathcal{B}(X, Y)$. The dimension of the range of $T \in \mathcal{F}(X, Y)$ is called the rank of T , denoted by $\text{rank}(T)$. For $x \in X$ and $f \in X^*$, denote by $x \otimes f$ the rank-one linear operator defined by $(x \otimes f)y = f(y)x = \langle y, f \rangle x$.

The following basic result can be proved as [7, Theorem 1.2]; we omit its proof here.

Theorem 2.1. *Let $\Phi : \mathcal{F}(X) \rightarrow \mathcal{F}(Y)$ be a bounded linear map. Then Φ is rank-nonincreasing if and only if Φ takes one of the following forms:*

- (1) *There are $A \in \mathcal{B}(X, Y)$ and $C \in \mathcal{B}(X^*, Y^*)$ such that $\Phi(T) = ATC^*|_Y$;*
- (2) *There are $A \in \mathcal{B}(X^*, Y)$ and $C \in \mathcal{B}(X, Y^*)$ such that $\Phi(T) = AT^*C^*|_Y$;*
- (3) *There is a bounded linear map $\varphi : \mathcal{F}(X) \rightarrow Y$ and a linear functional $f_0 \in Y^*$ such that $\Phi(T) = \varphi(T) \otimes f_0$;*
- (4) *There is a vector $x_0 \in Y$ and a bounded linear map $\psi : \mathcal{F}(X) \rightarrow Y^*$ such that $\Phi(T) = x_0 \otimes \psi(T)$.*

Theorem 2.2. *Let $\Phi : \mathcal{F}(X) \rightarrow \mathcal{F}(Y)$ be a bounded linear map. Then Φ is rank-preserving if and only if one of the following holds:*

- (1) *There exist injective linear operators $A \in \mathcal{B}(X, Y)$ and $C \in \mathcal{B}(X^*, Y^*)$ such that $\Phi(T) = ATC^*|_Y$.*
- (2) *There exist injective linear transformations $A \in \mathcal{B}(X^*, Y)$ and $C \in \mathcal{B}(X, Y^*)$ such that $\Phi(T) = AT^*C^*|_Y$.*

Proof. Since the range of Φ contains elements of rank > 1 , so Φ cannot take the forms in (3) and (4) of Theorem 2.1. Assume that Φ has the form in Theorem 2.1(1). Φ is rank-preserving implies that $\Phi(x \otimes f) = Ax \otimes Cf \neq 0$ if $x \otimes f \neq 0$. Therefore, both A and C are injective. Conversely, assume that both A and C are injective. For any $T \in \mathcal{F}(X)$, if $\text{rank}(T) = n$, then there are linearly independent subsets $\{x_i\}_{i=1}^n \subset X$ and $\{f_i\}_{i=1}^n \subset X^*$ such that $T = \sum_{i=1}^n x_i \otimes f_i$. The injectivity of A and C implies that $\{Ax_i\}_{i=1}^n$ and $\{Cf_i\}_{i=1}^n$ are linear independent subsets and hence $\Phi(T) = \sum_{i=1}^n Ax_i \otimes Cf_i$ has rank n , too. (2) can be proved similarly. \square

Notice that if both X and Y are reflexive, then the operator $C^*|_Y$ is in fact an operator from Y into X (or X^*), so we have

Corollary 2.3. *Let X and Y be reflexive Banach spaces and $\Phi : \mathcal{F}(X) \rightarrow \mathcal{F}(Y)$ be a bounded linear map. Then Φ is rank-nonincreasing if and only if Φ takes one of the following forms:*

- (1) *There are $A \in \mathcal{B}(X, Y)$ and $B \in \mathcal{B}(Y, X)$ such that $\Phi(T) = ATB$;*
- (2) *There are $A \in \mathcal{B}(X^*, Y)$ and $B \in \mathcal{B}(Y, X^*)$ such that $\Phi(T) = AT^*B$;*
- (3) *There is a bounded linear map $\varphi : \mathcal{F}(X) \rightarrow Y$ and a linear functional $f_0 \in Y^*$ such that $\Phi(T) = \varphi(T) \otimes f_0$;*
- (4) *There is a vector $x_0 \in Y$ and a bounded linear map $\psi : \mathcal{F}(X) \rightarrow Y^*$ such that $\Phi(T) = x_0 \otimes \psi(T)$.*

Particularly, for finite dimensional case, we have the following. Here T^t denotes the transpose of T .

Corollary 2.4. *A linear map $\Phi : M_n(\mathbb{F}) \rightarrow M_m(\mathbb{F})$ is rank-nonincreasing if and only if one of the following holds:*

- (1) *There are $m \times n$ matrix A and $n \times m$ matrix B such that $\Phi(T) = ATB$;*
- (2) *There are $m \times n$ matrix A and $n \times m$ matrix B such that $\Phi(T) = AT^tB$;*
- (3) *There are $m \times n$ matrices A_1, \dots, A_r , vectors $x_1, \dots, x_r \in \mathbb{F}^n$ and $f_0 \in \mathbb{F}^m$ such that $\Phi(T) = \sum_{i=1}^r A_i T(x_i \otimes f_0)$;*
- (4) *There are $n \times m$ matrices B_1, \dots, B_r , vectors $x_0 \in \mathbb{F}^n$ and $f_1, \dots, f_r \in \mathbb{F}^m$ such that $\Phi(T) = \sum_{i=1}^r (x_0 \otimes f_i) T B_i$.*

Proof. By Theorem 2.1, we need only prove (3) and (4). If Φ has the form in Theorem 2.1(3), then there exist a linear map $\varphi : M_n(\mathbb{F}) \rightarrow \mathbb{F}^m$ and a vector $f_0 \in \mathbb{F}^m$ such that $\Phi(T) = \varphi(T) \otimes f_0$. So there are m linear functionals φ_i on $M_n(\mathbb{F})$ such that $\varphi(T) = (\varphi_1(T), \dots, \varphi_m(T)) = \sum_{k=1}^m \varphi_k(T) e_k$, where $\{e_k\}_{k=1}^m$ is the standard basis

of \mathbb{F}^m . For every k , there exist vectors $x_{k1}, \dots, x_{kr_k}, f_{k1}, \dots, f_{kr_k} \in \mathbb{F}^n$ such that $\varphi_k(T) = \sum_{i=1}^{r_k} \langle T x_{ki}, f_{ki} \rangle$. Therefore, we have

$$\begin{aligned} \Phi(T) &= \left[\sum_{k=1}^m \sum_{i=1}^{r_k} \langle T x_{ki}, f_{ki} \rangle e_k \right] \otimes f_0 \\ &= \sum_{k=1}^m \left[\sum_{i=1}^{r_k} (e_k \otimes f_{ki}) \right] T(x_{ki} \otimes f_0) = \sum_{k=1}^m A_k T(x_{ki} \otimes f_0) \end{aligned}$$

with $A_k = \sum_{i=1}^{r_k} (e_k \otimes f_{ki})$. (4) is treated similarly. \square

Corollary 2.5. *Let $\Phi : M_n(\mathbb{F}) \rightarrow M_n(\mathbb{F})$ be a linear map. Then Φ is rank-preserving if and only if one of the following holds:*

- (1) *There exist invertible matrices $A, B \in M_n(\mathbb{F})$ such that $\Phi(T) = ATB$ for all $T \in M_n(\mathbb{F})$;*
- (2) *There exist invertible matrices $A, B \in M_n(\mathbb{F})$ such that $\Phi(T) = AT^t B$ for all $T \in M_n(\mathbb{F})$, where T^t is the transpose of T .*

3. Completely rank-nonincreasing linear maps

In [4] Hadwin and Larson introduced the notion of completely rank-nonincreasing linear maps. If X is a Banach space, let X^n denote a direct sum of n copies of X (with any norm that gives the topology of coordinatewise norm convergence). In the case in which X is a Hilbert space, we give X^n the ℓ^2 -norm. We then have, for any $n \in \mathbb{N}$, that $B(X^n)$ is isomorphic to $\mathcal{M}_n(B(X))$, the set of all $n \times n$ matrices with entries in $B(X)$. If $\mathcal{S} \subset B(X)$, Y is a Banach space and $\Phi : \mathcal{S} \rightarrow B(Y)$ is linear, we define $\Phi_n : \mathcal{M}_n(\mathcal{S}) \rightarrow \mathcal{M}_n(B(Y))$ by

$$\Phi_n((s_{ij})) = (\varphi(s_{ij})).$$

We say that Φ is *completely rank-nonincreasing* if, for every $n \in \mathbb{N}$, Φ_n is rank-nonincreasing.

We call the map Φ a *skew-compression* if there are operators $A \in B(X, Y)$, $C \in B(Y, X)$ such that, for every $S \in \mathcal{S}$,

$$\Phi(S) = ASC.$$

We also call Φ a *similarity* if there is an invertible operator B such that, for every $S \in \mathcal{S}$,

$$\Phi(S) = BSB^{-1}.$$

Here are two conjectures based on results and conjectures in [5] where affirmative answers were obtained in the Hilbert space case.

Conjecture 1. Suppose X is a Banach space and $1 \in \mathcal{S} \subset B(X)$ is a linear subspace and $\Phi : \mathcal{S} \rightarrow \mathcal{B}(X)$ is a linear map with $\Phi(1) = 1$. The following are equivalent:

- (1) Φ is a point-strong limit of similarities.
- (2) $\Phi|_{\mathcal{S} \cap \mathcal{F}(X)}$ is a point-weak limit of skew-compressions.

Conjecture 2. Suppose X and Y are Banach spaces, $\mathcal{S} \subset B(X)$ is a linear subspace and $\Phi : \mathcal{S} \rightarrow \mathcal{B}(Y)$ is a linear map. The following are equivalent:

- (1) Φ is a point-strong limit of skew-compressions.
- (2) $\Phi|_{\mathcal{S} \cap \mathcal{F}(X)}$ is a point-weak limit of skew-compressions.

If, as in the Hilbert space case [5], the above two conjectures are true, then the problem of characterizing point-strong limits of similarities is reduced to the case in which the Banach spaces involved are finite-dimensional. Since norms are equivalent in finite dimensions, if the above conjectures were true, then the conjecture in [4] would be equivalent to the following conjecture.

Conjecture 3. Suppose X and Y are Banach spaces, $\mathcal{S} \subset B(X)$ is a linear subspace and $\Phi : \mathcal{S} \rightarrow B(Y)$ is a linear map. The following are equivalent:

- (1) Φ is a point-strong limit of skew-compressions.
- (2) Φ is completely rank-nonincreasing.

It is clear that the implication (1) \Rightarrow (2) in Conjecture 3 is true. Hence the meat of Conjecture 3 is that Φ being completely rank-nonincreasing implies that Φ is a point-strong limit of skew-compressions. Note that Conjecture 3 easily implies Conjecture 2.

We will focus mainly in the finite-dimensional case. In this case, the limits of similarity problem is equivalent to the characterization of norm closures of joint similarity orbits. Suppose $\vec{T} = (T_1, \dots, T_k)$ is a k -tuple of $n \times n$ (real or) complex matrices. The joint similarity orbit of \vec{T} is the set

$$\mathcal{S}(\vec{T}) = \{(A^{-1}T_1A, \dots, A^{-1}T_kA) : A \text{ is invertible}\}.$$

The finite-dimensional version of Conjecture 3 implies the following conjecture from [4]. The key reason is that in finite dimensions a unital limit of skew-compressions is a limit of similarities. To see this suppose $1 \in \mathcal{S} \subset \mathcal{M}_n$ and $\Phi : \mathcal{S} \rightarrow \mathcal{M}_n$ is a limit of skew-compressions and $\Phi(1) = 1$. If $\Phi(S) = \lim_{m \rightarrow \infty} A_m S B_m$ for each

$S \in \mathcal{S}$, then $A_m B_m \rightarrow 1$, which means eventually A_m and B_m are both invertible, so

$$\Phi(S) = \lim_{m \rightarrow \infty} (A_m B_m)^{-1} A_m S B_m = \lim_{m \rightarrow \infty} B_m^{-1} S B_m$$

is a limit of similarities.

Conjecture 4. $\vec{S} \in \mathcal{S}(\vec{T})^-$ if and only if the mapping

$$p(\vec{T}) \mapsto p(\vec{S}),$$

for each polynomial $p(t_1, \dots, t_k)$ in free variables t_1, \dots, t_k is well-defined and extends to a completely rank-nonincreasing linear map from the algebra generated by T_1, \dots, T_k to the algebra generated by S_1, \dots, S_k .

In [2] Curto and Herrero conjectured that Conjecture 4 holds with “completely rank-nonincreasing” replaced with “rank nonincreasing”. However, it was shown in [4] that their conjecture is false.

If Conjecture 4 is true, then every unital completely rank-nonincreasing linear map should be a limit of similarities. The next result is positive evidence.

Theorem 3.1. *If $\Phi : \mathcal{M} \subseteq M_n(\mathbb{F}) \rightarrow M_n(\mathbb{F})$ is completely rank-nonincreasing (preserving) linear map and $\Phi(I_n) = I_n$, then Φ can be extended to a completely rank-nonincreasing (preserving) algebraic homomorphism on the algebra generated by \mathcal{M} .*

Proof. Denote $I = I_n \in \mathcal{M}$. For any $T_1, \dots, T_k \in M_n(\mathbb{F})$ let

$$\mathbf{T} = \begin{pmatrix} T_1 & -I & 0 & \cdots & 0 & 0 \\ 0 & T_2 & -I & \cdots & 0 & 0 \\ 0 & 0 & T_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & T_{k-1} & -I \\ 0 & 0 & 0 & \cdots & 0 & T_k \end{pmatrix},$$

$$\mathbf{A} = \begin{pmatrix} -I & 0 & 0 & \cdots & 0 & 0 \\ -T_2 & -I & 0 & \cdots & 0 & 0 \\ -T_3 T_2 & -T_3 & -I & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -T_{k-1} \cdots T_2 & -T_{k-1} \cdots T_3 & -T_{k-1} \cdots T_4 & \cdots & -I & 0 \\ T_k \cdots T_2 & T_k \cdots T_3 & T_k \cdots T_4 & \cdots & T_k & I \end{pmatrix},$$

and

$$\mathbf{B} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & I \\ I & 0 & 0 & \cdots & 0 & T_1 \\ 0 & I & 0 & \cdots & 0 & T_2 T_1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & T_{k-2} T_{k-3} \cdots T_1 \\ 0 & 0 & 0 & \cdots & I & T_{k-1} T_{k-2} \cdots T_1 \end{pmatrix}.$$

Then a straight computation shows that

$$\mathbf{A}\mathbf{T}\mathbf{B} = \begin{pmatrix} I & 0 & 0 & \cdots & 0 & 0 \\ 0 & I & 0 & \cdots & 0 & 0 \\ 0 & 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & I & 0 \\ 0 & 0 & 0 & \cdots & 0 & T_k \cdots T_2 T_1 \end{pmatrix}.$$

It is clear that both \mathbf{A} and \mathbf{B} are invertible, so we have

$$\text{rank}(T_k \cdots T_2 T_1) = \text{rank}(\mathbf{T}) - (k-1)n.$$

If $T_1, \dots, T_k \in \mathcal{M}$, then $\mathbf{T} \in \mathcal{M} \otimes M_k(\mathbb{F})$ and we have, by above notice,

$$\begin{aligned} \text{rank}(\Phi(T_k) \cdots \Phi(T_2)\Phi(T_1)) &= \text{rank}(\Phi_k(\mathbf{T})) - (k-1)n \\ &\leq \text{rank}(\mathbf{T}) - (k-1)n = \text{rank}(T_k \cdots T_2 T_1) \end{aligned}$$

since Φ is unital and completely rank-nonincreasing. Namely, the rank of the product of images of a unital completely rank-nonincreasing linear map at some matrices is not larger than the rank of the corresponding product of these matrices.

Now assume that $T \in M_n(\mathbb{F})$ has the form

$$T = \sum_{r=1}^m S_{r1} \cdots S_{rk_r} \quad \text{where } S_{rs_r} \in \mathcal{M} \text{ for all } r = 1, \dots, m \text{ and } s_r = 1, \dots, k_r.$$

Let $\mathbf{W}_{rs_r} = (W_{ij}^{(rs_r)}) \in \mathcal{M} \otimes M_m(\mathbb{F})$ with $W_{rr}^{(rs_r)} = S_{rs_r}$, $W_{ii}^{(rs_r)} = I$ if $i \neq r$ and $W_{ij}^{(rs_r)} = 0$ if $i \neq j$; $r = 1, \dots, m$ and $s_r = 1, \dots, k_r$. Let $\mathbf{A} = (A_{ij})$ with $A_{ij} = I$ for all (i, j) . Then

$$\mathbf{A}\mathbf{W}_{11} \cdots \mathbf{W}_{1k_1} \mathbf{W}_{21} \cdots \mathbf{W}_{(m-1)k_{(m-1)}} \mathbf{W}_{m1} \cdots \mathbf{W}_{mk_m} \mathbf{A} = \mathbf{V} = (V_{ij})$$

with $V_{ij} = T = \sum_{r=1}^m S_{r1} \cdots S_{rk_r}$ for every (i, j) . It follows that

$$\begin{aligned} &\text{rank} \left(\sum_{r=1}^m S_{r1} \cdots S_{rk_r} \right) \\ &= \text{rank}(\mathbf{A}\mathbf{W}_{11} \cdots \mathbf{W}_{1k_1} \mathbf{W}_{21} \cdots \mathbf{W}_{(m-1)k_{(m-1)}} \mathbf{W}_{m1} \cdots \mathbf{W}_{mk_m} \mathbf{A}). \end{aligned}$$

Now, since $\Phi_m : \mathcal{M} \otimes M_m(\mathbb{F}) \rightarrow M_n(\mathbb{F}) \otimes M_m(\mathbb{F})$ is completely rank-nonincreasing, we have

$$\begin{aligned} & \text{rank}(\Phi_m(\mathbf{A})\Phi_m(\mathbf{W}_{11})\cdots\Phi_m(\mathbf{W}_{1k_1})\Phi_m(\mathbf{W}_{21})\cdots\Phi_m(\mathbf{W}_{(m-1)k_{(m-1)}}) \\ & \quad \cdot \Phi_m(\mathbf{W}_{m1})\cdots\Phi_m(\mathbf{W}_{mk_m})\Phi_m(\mathbf{A})) \\ & \leq \text{rank}(\mathbf{A}\mathbf{W}_{11}\cdots\mathbf{W}_{1k_1}\mathbf{W}_{21}\cdots\mathbf{W}_{(m-1)k_{(m-1)}}\mathbf{W}_{m1}\cdots\mathbf{W}_{mk_m}\mathbf{A}), \end{aligned}$$

which implies that

$$\text{rank}\left(\sum_{r=1}^m \Phi(S_{r1})\cdots\Phi(S_{rk_r})\right) \leq \text{rank}\left(\sum_{r=1}^m S_{r1}\cdots S_{rk_r}\right).$$

Let $\widehat{\mathcal{M}}$ be the algebra generated by \mathcal{M} and let $\widehat{\Phi} : \widehat{\mathcal{M}} \rightarrow M_n(\mathbb{F})$ be a linear map determined by

$$\widehat{\Phi}\left(\sum_{r=1}^m S_{r1}\cdots S_{rk_r}\right) = \left(\sum_{r=1}^m \Phi(S_{r1})\cdots\Phi(S_{rk_r})\right).$$

$\widehat{\Phi}$ is well defined, for if $\sum_{r=1}^m S_{r1}\cdots S_{rk_r} = 0$, then

$$0 \leq \text{rank}\left(\sum_{r=1}^m \Phi(S_{r1})\cdots\Phi(S_{rk_r})\right) \leq \text{rank}\left(\sum_{r=1}^m S_{r1}\cdots S_{rk_r}\right) = 0$$

which forces $\sum_{r=1}^m \Phi(S_{r1})\cdots\Phi(S_{rk_r}) = 0$. It is clear that $\widehat{\Phi}$ extends Φ and is an algebraic homomorphism. $\widehat{\Phi}$ is also completely rank-nonincreasing because $\widehat{\Phi}_m = \widehat{\Phi}_m$.

If Φ is completely rank-preserving, “ \leq ” between “rank” in the argument above are exactly “=”, so $\widehat{\Phi}$ is completely rank-preserving in this case. \square

The next result shows that, in the finite-dimensional case, Conjecture 4 and Conjecture 2 are equivalent.

Theorem 3.2. Suppose \mathcal{F} is a field, $\mathcal{S} \subset \mathcal{M}_n(\mathcal{F})$ is a linear subspace and $\varphi : \mathcal{S} \rightarrow \mathcal{M}_n(\mathcal{F})$ is a completely rank-nonincreasing linear map. Let

$$\mathcal{A} = \left\{ \begin{pmatrix} \lambda I & A \\ 0 & \lambda I \end{pmatrix} : \lambda \in \mathcal{F}, A \in \mathcal{S} \right\},$$

and define $\Phi : \mathcal{A} \rightarrow \mathcal{M}_2(\mathcal{M}_n(\mathcal{F})) = \mathcal{M}_{2n}(\mathcal{F})$ by

$$\Phi\left(\begin{pmatrix} \lambda I & A \\ 0 & \lambda I \end{pmatrix}\right) = \begin{pmatrix} \lambda I & \varphi(A) \\ 0 & \lambda I \end{pmatrix}.$$

Then Φ is a unital algebra homomorphism and Φ is completely rank-nonincreasing. Furthermore, if $\mathcal{F} = \mathbb{R}$ or $\mathcal{F} = \mathbb{C}$, then Φ is a limit of similarities if and only if φ is a limit of skew-compressions.

Proof. Note that $\mathcal{J} = \left\{ \begin{pmatrix} 0 & S \\ 0 & 0 \end{pmatrix} : S \in \mathcal{S} \right\}$ is the Jacobson radical of \mathcal{A} , and, for each $m \in \mathcal{N}$, $\mathcal{M}_m(\mathcal{J})$ is the Jacobson radical of $\mathcal{M}_m(\mathcal{A})$. Note also, for every $m \in$

\mathbb{N} , Φ_m is a unital homomorphism, which implies that Φ_m maps invertible elements to invertible elements. Since φ is completely rank-nonincreasing, it is clear that $\Phi|_{\mathcal{J}}$ is completely rank-nonincreasing.

Consider an $m \times m$ matrix $T = (T_{ij})$ in $\mathcal{M}_m(\mathcal{A})$. We can write $T_{ij} = \lambda_{ij} I_{2n} + C_{ij}$, where $C_{ij} \in \mathcal{J}$. Using row and column operations we can find invertible matrices $(\alpha_{ij} I_{2n})$, $(\beta_{ij} I_{2n})$ (with $\alpha_{ij}, \beta_{ij} \in \mathcal{F}$) such that

$$(\alpha_{ij} I_{2n})(\lambda_{ij} I_{2n})(\beta_{ij} I_{2n}) = \text{diag}(I_{2n}, \dots, I_{2n}, 0, \dots, 0),$$

with exactly k copies of I_{2n} . The remarks about $\mathcal{M}_m(\mathcal{J})$ being the Jacobson radical of $\mathcal{M}_m(\mathcal{A})$ implies that the upper left $k \times k$ corner of

$$(\alpha_{ij} I_{2n})T(\beta_{ij} I_{2n})$$

is invertible. Hence, there are invertible matrices $A, B \in \mathcal{M}_m(\mathcal{A})$ such that

$$A(\alpha_{ij} I_{2n})T(\beta_{ij} I_{2n})B = \begin{pmatrix} I & 0 \\ 0 & D \end{pmatrix}$$

where I is a direct sum of k copies of I_{2n} and $D \in \mathcal{M}_{m-k}(\mathcal{J})$. However,

$$\Phi_m \left(\begin{pmatrix} I & 0 \\ 0 & D \end{pmatrix} \right) = \begin{pmatrix} I & 0 \\ 0 & \Phi_{m-k}(D) \end{pmatrix}.$$

Since $\Phi|_{\mathcal{J}}$ is completely rank-nonincreasing and Φ_m sends invertible elements to invertible elements, we have

$$\begin{aligned} \text{rank } \Phi_m(T) &= \text{rank } \Phi_m \left(\begin{pmatrix} I & 0 \\ 0 & D \end{pmatrix} \right) = 2nk + \text{rank } \Phi_{m-k}(D) \\ &\leq 2nk + \text{rank } D = \text{rank } \begin{pmatrix} I & 0 \\ 0 & D \end{pmatrix} = \text{rank } T. \end{aligned}$$

Hence Φ is completely rank-nonincreasing.

Next suppose the field \mathcal{F} is \mathbb{R} or \mathbb{C} . It is obvious that if Φ is a limit of similarities, then, restricting Φ to \mathcal{J} , φ is limit of skew-compressions. Conversely, suppose $\varphi(S) = \lim_{m \rightarrow \infty} A_m S B_m$ for every S in \mathcal{S} . Since the set of invertible $n \times n$ matrices is dense in the set of all matrices, we can assume that A_m and B_m are invertible for each m . If we define $V_m = \begin{pmatrix} A_m & 0 \\ 0 & B_m^{-1} \end{pmatrix}$, we see that $\Phi(A) = \lim_{m \rightarrow \infty} V_m A V_m^{-1}$ for every $A \in \mathcal{A}$. \square

We next consider some more conjectures concerning equivalence of linear spaces of matrices. Suppose \mathcal{S} is a set of $m \times n$ matrices. We can make \mathcal{S} into a set of $k \times k$ matrices ($k \geq \max(m, n)$) by adding rows of zeros or columns of zeros to each matrix in \mathcal{S} . We extend the notion of equivalence to include multiplying \mathcal{S} on the left and on the right by invertible matrices. This notion of equivalence preserves many properties, e.g., reflexivity. One important property that is preserved is rank. In fact, if \mathcal{T} is equivalent to \mathcal{S} and $\Phi : \mathbb{S} \rightarrow \mathcal{T}$ is the mapping that adds (or deletes)

zero rows or columns and multiplies on the left and right by invertible matrices, then Φ is completely rank-preserving. We conjecture that the converse is true.

Conjecture 5. *If \mathcal{S} is a linear subspace of \mathcal{M}_n and $\Phi : \mathcal{S} \rightarrow \mathcal{M}_n$ is a completely rank-preserving linear map, then Φ is a skew-compression with invertible left and right multipliers. (This would mean \mathcal{S} is equivalent to $\Phi(\mathcal{S})$.)*

The next conjecture is equivalent to the last one.

Conjecture 6. *Suppose \mathcal{A} is a unital algebra of $n \times n$ matrices and $\Phi : \mathcal{A} \rightarrow \mathcal{M}_n$ is a unital linear algebra homomorphism that is completely rank-preserving. Then Φ is a similarity.*

This last conjecture was proved by Barria and Herrero (see [6]) in the case in which \mathcal{A} is the unital algebra generated by a single matrix, with “completely rank-preserving” replaced with “rank-preserving”.

If the above two conjectures are true, then if $\Phi : \mathcal{S} \rightarrow \mathcal{M}_n$ is an invertible linear map and Φ and Φ^{-1} are limits of skew-compressions, then Φ is a skew-compression with invertible multipliers. Equivalently, if \mathcal{A} and \mathcal{B} are unital subalgebras of \mathcal{M}_n and $\Phi : \mathcal{A} \rightarrow \mathcal{B}$ is a unital algebra isomorphism such that Φ and Φ^{-1} are both limits of similarities, then Φ is a similarity. We cannot prove this, but we can prove a weaker result. The following result was proved at a lunch attended by a large group of mathematicians too numerous to name; but, considering this paper’s dedication, we want to at least mention that Heydar Radjavi was a participant.

Lemma 3.1. *Suppose \mathcal{S} is a linear subspace of \mathcal{M}_n and $\Phi : \mathcal{S} \rightarrow \mathcal{T} \subset \mathcal{M}_n$ is an invertible linear map so that Φ and Φ^{-1} are skew-compressions. Then there are invertible matrices V and W such that $\Phi(S) = VSW$ for every $S \in \mathcal{S}$.*

Proof. We suppose $\Phi(S) = ASB$ and $\Phi^{-1}(T) = CTD$ for all $S \in \mathcal{S}$ and $T \in \mathcal{T}$. Then $S = XSY$ for all $S \in \mathcal{S}$, where $X = CA$ and $Y = BD$. We will show that X (and hence A) is injective on $M = sp(\cup_{S \in \mathcal{S}} \text{ran } S)$. Hence we can redefine A on M^\perp so that A is invertible. A similar argument works for B by taking adjoints. Let $N = sp(\cup_{S \in \mathcal{S}} \text{ran } SY)$. Clearly $N \subset M$, but

$$X(N) = sp \cup_{S \in \mathcal{S}} \text{ran}(XSY) = M.$$

It follows that $\dim M = \dim N$, and thus $M = N$ and $X(M) = M$. Hence $X|_M$ is injective. \square

Note that the last two conjectures make sense for matrices over an arbitrary field, and a solution or counterexample in this more general setting would be welcomed.

We conclude this section with some positive results. The first concerns finite rank operators.

Theorem 3.3. *Suppose X and Y are Banach spaces and let $\Phi : \mathcal{F}(X) \rightarrow \mathcal{F}(Y)$ be a bounded linear map. The following statements are equivalent.*

- (1) Φ is completely rank-nonincreasing.
- (2) Φ_2 is rank-nonincreasing.
- (3) There exist $A \in \mathcal{B}(X, Y)$, $C \in \mathcal{B}(X^*, Y^*)$ such that $\Phi(T) = ATC^*|_Y$ for all $T \in \mathcal{F}(X)$.
- (4) There exist $A \in \mathcal{B}(X, Y)$ and a net $\{B_\lambda : \lambda \in A\} \subseteq \mathcal{B}(Y, X)$ such that for every $T \in \mathcal{F}(X)$, $\Phi(T) = \lim_\lambda ATB_\lambda(WOT)$.

Proof. It is obvious that (4) \Rightarrow (1) \Rightarrow (2).

(3) \Rightarrow (4). Suppose (3). For any finite subset $\lambda = \{y_i; f_i : i = 1, \dots, n\}$, where $y_i \in Y$ and $f_i \in X^*$, there exists a $B_\lambda \in \mathcal{B}(Y, X)$ such that $\langle f_i, B_\lambda y_i \rangle = \langle f_i, C^* y_i \rangle$, $i = 1, \dots, n$. Let A be the set of all such finite set λ and partially ordered by inclusion of sets, i.e., $\lambda_1 \succcurlyeq \lambda_2$ if and only if $\lambda_1 \supseteq \lambda_2$. Take B_λ for each $\lambda \in A$, then for every $y \in Y$ and $f \in X^*$ we have $\lim_\lambda \langle B_\lambda y, f \rangle = \langle f, Cy \rangle$. So, for every $T \in \mathcal{F}(X)$, $\Phi(T) = \lim_\lambda ATB_\lambda(WOT)$.

(2) \Rightarrow (3). Assume that $\Phi \neq 0$ is completely rank-nonincreasing, then by Theorem 2.1, Φ takes one of the forms of Theorem 2.1(1)–(4). We must prove Φ can take only the form of Theorem 2.1(1). We may assume that $\dim X \geq 2$.

Assume that Φ has the form of Theorem 2.1(2), i.e., $\Phi(T) = AT^*C^*|_Y$ for every T . Take $x_i \in X$ and $f_i \in X^*$, $i = 1, 2$. Then $T = \begin{pmatrix} x_1 \otimes f_1 & x_1 \otimes f_2 \\ x_2 \otimes f_1 & x_2 \otimes f_2 \end{pmatrix} \in \mathcal{F}(X) \otimes M_2$

M_2 is of rank one and $\Phi_2(T) = \begin{pmatrix} Af_1 \otimes Cx_1 & Af_2 \otimes Cx_1 \\ Af_1 \otimes Cx_2 & Af_2 \otimes Cx_2 \end{pmatrix}$.

If $\text{rank}(A) \geq 2$ or $\text{rank}(C) \geq 2$, for instance, say $\text{rank}(A) \geq 2$, then there exist $f_1, f_2 \in X^*$ such that Af_1 is linearly independent to Af_2 . It is easy to check that $\text{rank}(\Phi_2(T)) \geq 2$, contradicting to that Φ is completely rank-nonincreasing. If both A and C have rank 1, then $A = u \otimes v$ for some $u \in Y$, $v \in X^{**}$ and $C = g \otimes h$ for some $g \in Y^*$, $h \in X^*$. Thus for any rank-one $x \otimes f \in \mathcal{F}(X)$, we have

$$\Phi(x \otimes f) = Af \otimes Cx = v(f)u \otimes h(x)g = (u \otimes h)(x \otimes f)(v \otimes g),$$

and hence Φ has the form of Theorem 2.1(1).

Assume that Φ has the form of Theorem 2.1(3), that is, $\Phi(T) = \varphi(T) \otimes f_0$. For any rank-one element $S = \begin{pmatrix} x_1 \otimes f_1 & x_1 \otimes f_2 \\ x_2 \otimes f_1 & x_2 \otimes f_2 \end{pmatrix} \in \mathcal{F}(X) \otimes M_2$,

$$\Phi_2(S) = \begin{pmatrix} \varphi(x_1 \otimes f_1) \otimes f_0 & \varphi(x_1 \otimes f_2) \otimes f_0 \\ \varphi(x_2 \otimes f_1) \otimes f_0 & \varphi(x_2 \otimes f_2) \otimes f_0 \end{pmatrix}$$

is rank-one if and only if $\begin{pmatrix} \varphi(x_1 \otimes f_1) \\ \varphi(x_2 \otimes f_1) \end{pmatrix}$ and $\begin{pmatrix} \varphi(x_1 \otimes f_2) \\ \varphi(x_2 \otimes f_2) \end{pmatrix}$ are linearly dependent. Take x_1 and f_1 so that $\varphi(x_1 \otimes f_1) \neq 0$. Then

$$\begin{pmatrix} \varphi(x_1 \otimes f) \\ \varphi(x \otimes f) \end{pmatrix} = \alpha(x, f) \begin{pmatrix} \varphi(x_1 \otimes f_1) \\ \varphi(x \otimes f_1) \end{pmatrix}$$

holds for all $x \in X$ and $f \in X^*$. It is easy to see that $\alpha(x, f) \in \mathbb{F}$ is in fact independent of x , so $\alpha(x, f) = \alpha(f)$ and $\alpha \in X^{**}$. Let $Ax = \varphi(x \otimes f_1)$, then $A \in \mathcal{B}(X, Y)$. It follows that $\Phi(x \otimes f) = \alpha(f)Ax \otimes f_0 = A(x \otimes f)(\alpha \otimes f_0)$ and hence $\Phi(T) = AT(\alpha \otimes f_0)$ for all $T \in \mathcal{F}(X)$, which is the form desired.

Assume that Φ has the form of Theorem 2.1(4), i.e., $\Phi(T) = x_0 \otimes \psi(T)$. Let S be the rank-one linear transformation as above.

$$\Phi_2(S) = \begin{pmatrix} x_0 \otimes \psi(x_1 \otimes f_1) & x_0 \otimes \psi(x_1 \otimes f_2) \\ x_0 \otimes \psi(x_2 \otimes f_1) & x_0 \otimes \psi(x_2 \otimes f_2) \end{pmatrix}$$

is rank-one if and only if $\begin{pmatrix} \psi(x_1 \otimes f_1) \\ \psi(x_1 \otimes f_2) \end{pmatrix}$ and $\begin{pmatrix} \psi(x_2 \otimes f_1) \\ \psi(x_2 \otimes f_2) \end{pmatrix}$ are linearly dependent. Take x_1 and f_1 so that $\psi(x_1 \otimes f_1) \neq 0$. It is clear that there exists a $\beta \in X^*$ such that $\begin{pmatrix} \psi(x \otimes f_1) \\ \psi(x \otimes f) \end{pmatrix} = \beta(x) \begin{pmatrix} \psi(x_1 \otimes f_1) \\ \psi(x_1 \otimes f) \end{pmatrix}$ holds for all $x \in X$ and $f \in X^*$. Denote $Cf = \psi(x_1 \otimes f)$ for every f in X^* , it is obvious that $C \in \mathcal{B}(X^*, Y^*)$. Now, we have $\Phi(x \otimes f) = x_0 \otimes \beta(x)Cf = (x_0 \otimes \beta)(x \otimes f)C^*$ and hence $\Phi(T) = (x_0 \otimes \beta)TC^*$ for all $T \in \mathcal{F}(X)$, as desired. \square

Corollary 3.1. *Let X and Y be reflexive Banach spaces over \mathbb{R} or \mathbb{C} . Suppose that $\Phi : \mathcal{F}(X) \rightarrow \mathcal{F}(Y)$ is a bounded linear map, the following statements are equivalent.*

- (1) Φ is completely rank-nonincreasing.
- (2) Φ_2 is rank-nonincreasing.
- (3) There exist $A \in \mathcal{B}(X, Y)$, $B \in \mathcal{B}(Y, X)$ such that $\Phi(T) = ATB$ for all $T \in \mathcal{F}(X)$.

For completely rank-preserving linear maps, we have

Corollary 3.2 *Let $\Phi : \mathcal{F}(X) \rightarrow \mathcal{F}(Y)$ be a bounded linear map. The following statements are equivalent.*

- (1) Φ is completely rank-preserving.
- (2) Φ_2 is rank-preserving.
- (3) There exist injective linear operators $A \in \mathcal{B}(X, Y)$, $C \in \mathcal{B}(X^*, Y^*)$ such that $\Phi(T) = ATC^*|_Y$ for all $T \in \mathcal{F}(X)$.

In particular, we have

Corollary 3.3. *Let $\Phi : M_n(\mathbb{F}) \rightarrow M_n(\mathbb{F})$ be a linear map, then the following statements are equivalent:*

- (1) Φ is completely rank-preserving.
- (2) Φ_2 is rank-preserving.
- (3) Φ is completely rank-nonincreasing and maps some invertible matrix to an invertible matrix.
- (4) Φ_2 is rank-nonincreasing and maps some invertible matrix to an invertible matrix.
- (5) There are invertible matrices $A, B \in M_n(\mathbb{F})$ such that $\Phi(T) = ATB$ for all $T \in M_n(\mathbb{F})$.

We can obtain some results in a purely algebraic setting.

Proposition 3.1. *Suppose \mathcal{F} is a field and \mathcal{A} is a unital semisimple subalgebra of $M_n(\mathcal{F})$ and suppose $\Phi : \mathcal{A} \rightarrow M_n(\mathcal{F})$ is a unital rank-preserving algebra homomorphism. Then Φ is a similarity.*

Proof. Since \mathcal{A} is semisimple and Artinian, \mathcal{A} , and hence $\Phi(\mathcal{A})$, is a direct sum of matrix algebras over division rings. Using a similarity, we can simultaneously diagonalize the central projections of $\Phi(\mathcal{A})$. Since Φ is rank-preserving, we can assume that $\Phi(P) = P$ for every minimal central projection $P \in \mathcal{A}$. This reduces to the case where \mathcal{A} is a matrix ring over a division ring \mathcal{D} . Again we can assume that $\Phi(P) = P$ for every minimal diagonal projection. From this point standard arguments show that Φ is a similarity. \square

4. k-rank-nonincreasing linear maps and a characterization of elementary operators

Let Φ be a linear map from $\mathcal{B}(X)$ into $\mathcal{B}(Y)$, where X and Y are Banach space over field $\mathbb{F}(= \mathbb{R} \text{ or } \mathbb{C})$. Recall that Φ is called an elementary operator if there exist operators $A_1, \dots, A_n \in \mathcal{B}(X, Y)$ and $B_1, \dots, B_n \in \mathcal{B}(Y, X)$ such that $\Phi(T) = \sum_{i=1}^n A_i T B_i$. The integer $r = \min\{n : \Phi(T) = \sum_{i=1}^n A_i T B_i\}$ is called the length of Φ . For $T \in \mathcal{B}(X)$, we denote $T^{(n)}$ the direct sum of n copies of T , which is an element in $\mathcal{B}(X^n)$ with diagonal entries T and all other entries 0. Let $\mathbf{A} =$

$$(A_1 \quad \cdots \quad A_n) \in \mathcal{B}(X^n, Y) \text{ and } \mathbf{B} = \begin{pmatrix} B_1 \\ \vdots \\ B_n \end{pmatrix} \in \mathcal{B}(Y, X^n). \text{ Then elementary operator } \Phi \text{ may also be written in } \Phi(T) = \mathbf{A} T^{(n)} \mathbf{B}.$$

Definition 4.1. We say a linear map $\Phi : \mathcal{F}(X) \rightarrow \mathcal{F}(Y)$ is k -rank-nonincreasing if $\text{rank}(\Phi(T)) \leq k(\text{rank}(T))$ for every T . Φ is completely k -rank-nonincreasing if Φ_n is k -rank-nonincreasing for every positive integer n .

For Hilbert space case, the first author and Larson [4] give a characterization of elementary operators in terms of rank function. They show that if H and K are Hilbert spaces, then a linear map $\Phi : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$ is an elementary operator if and only if Φ is $(\sigma\text{-w})$ – $(\sigma\text{-w})$ continuous, completely bounded and completely k -rank-nonincreasing. Recall that Φ is called completely bounded if $\|\Phi\|_{\text{cb}} = \sup_n \|\Phi_n\| < \infty$. In this section we discuss the k -rank-nonincreasing linear maps and give a characterization of elementary operators from $\mathcal{B}(X)$ into $\mathcal{B}(Y)$ in a much stronger version by omitting the “completely boundedness” assumption. To do this, we need first discuss the completely k -rank-nonincreasing linear maps from $\mathcal{F}(X)$ into $\mathcal{F}(Y)$ in some details.

Let Z be a linear space over \mathbb{F} , $\Psi : Z \rightarrow M_N(\mathbb{F})$ a linear map. Then there are N^2 linear functionals $\psi_{ij} : Z \rightarrow \mathbb{F}$ such that $\Psi(z) = (\psi_{ij}(z))$. Let $M_N(Z)$ be the linear space consisting of all $N \times N$ matrices with entries the elements in Z , and define a linear functional $\widehat{\Psi}$ on $M_N(Z)$ by $\widehat{\Psi}((z_{ij})) = \frac{1}{N} \sum_{i,j} \psi_{ij}(z_{ij})$. One can recover Ψ from $\widehat{\Psi}$ by $\psi_{ij}(x) = N\widehat{\Psi}(xE_{ij})$, here E_{ij} is $N \times N$ matrix with a 1 in the (i, j) -entry and 0's in the other entries. Similar to the Hilbert space case [4] we have the following lemma.

Lemma 4.1. Let $\Phi : \mathcal{F}(X) \rightarrow M_N(\mathbb{F})$ be a linear map, then

- (1) Φ is completely k -rank-nonincreasing if and only if $\widehat{\Phi}$ is.
- (2) Φ is an elementary operator of length r if and only if $\widehat{\Phi}$ is.
- (3) $\Phi(T) = \lim_{\lambda} A_{\lambda} T B_{\lambda}$ (WOT or SOT) pointwise if and only if $\widehat{\Phi}((T_{ij})) = \lim_{\lambda} C_{\lambda}((T_{ij})) D_{\lambda}$ (WOT or SOT) pointwise.

Theorem 4.1. Let $\Phi : \mathcal{F}(X) \rightarrow M_N(\mathbb{F})$ be a bounded linear map, then

- (1) Φ is completely k -rank-nonincreasing;
- (2) $\Phi_{(k+1)N^2}$ is k -rank-nonincreasing;
- (3) There exist $A_1, \dots, A_r \in \mathcal{B}(X, \mathbb{F}^N)$ and $C_1, \dots, C_r \in \mathcal{B}(X^*, \mathbb{F}^N)$ with $r \leq k$ such that $\Phi(T) = \sum_{i=1}^r A_i T C_i^*|_X$.

Proof. The “(3) \Rightarrow (1) \Rightarrow (2)” part is obvious, in fact, it is easy to see from Theorem 3.1 that every linear map of the form $\Phi(T) = \sum_{i=1}^r A_i T C_i^*|_X$ is completely r -rank-nonincreasing. For “(1) \Rightarrow (3)” part, assume that Φ is completely k -rank-nonincreasing. By Lemma 4.1 we may assume $N = 1$, that is, Φ is a linear functional. We have to prove that there exist $r \leq k$, vectors $\xi = (u_1 \ \dots \ u_r)^t \in (X^{**})^r$ and $\eta = (g_1 \ \dots \ g_r)^t \in (X^*)^r$ such that $\Phi(T) = \sum_{i=1}^r \langle T u_i, g_i \rangle = \langle T^{(k)} \xi, \eta \rangle$. Assume

that Φ is not completely $(k-1)$ -rank-nonincreasing (otherwise we consider $r = \min\{l: l \leq k, \Phi \text{ is completely } l\text{-rank-nonincreasing}\}$). Then there is a rank-one operator $S = (S_{ij}) \in M_n(\mathcal{F}(X))$ for some $n \geq k$ such that $\text{rank}(\Phi(S_{ij})) = k$. By taking a k -rank submatix of $(\Phi(S_{ij}))$ we see that we may take $n = k$ and hence $(\Phi(S_{ij})) \in M_k(\mathbb{F})$ is invertible (this also makes clear that $(2) \Rightarrow (1)$ for case $N = 1$). Thus there exist invertible $k \times k$ matrices (c_{ij}) and (d_{ij}) such that $(c_{ij})(\Phi(S_{ij}))(d_{ij}) = I_k$. Let $C = I \otimes (c_{ij})$ and $D = I \otimes (d_{ij})$, then CSD is still rank-one and it is easily checked that $\Phi_k(CSD) = I_k$. So, we may take rank-one operator $S = (S_{ij}) \in M_k(\mathcal{F}(X))$ such that $\Phi_k(S) = I_k$, that is, there exist vectors $\{x_i\}_{i=1}^k \subset X$ and linear functionals $\{f_i\}_{i=1}^k \subset X^*$ such that $\Phi(x_i \otimes f_j) = \delta_{ij}$. Now for any $x_{k+1} = x \in X$ and $f_{k+1} = f \in X^*$, $R = (x_i \otimes f_j)_{(k+1) \times (k+1)}$ is of rank-one and hence $\Phi_{k+1}(R)$ has rank not greater than k . Since

$$\Phi_{k+1}(R) = \begin{pmatrix} 1 & 0 & \cdots & 0 & \Phi(x_1 \otimes f) \\ 0 & 1 & \cdots & 0 & \Phi(x_2 \otimes f) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \Phi(x_k \otimes f) \\ \Phi(x \otimes f_1) & \Phi(x \otimes f_2) & \cdots & \Phi(x \otimes f_k) & \Phi(x \otimes f) \end{pmatrix},$$

we must have

$$\Phi(x \otimes f) = \sum_{i=1}^k \Phi(x \otimes f_i) \Phi(x_i \otimes f)$$

holds for all $x \in X$ and $f \in X^*$. Notice that $\Phi(x \otimes f)$ is a bounded bilinear form, there exists a bounded linear operator $G \in \mathcal{B}(X^*)$ such that $\Phi(x \otimes f) = \langle x, Gf \rangle$ ($= (Gf)(x)$). So we have

$$\Phi(x \otimes f) = \sum_{i=1}^k \langle x, Gf_i \rangle \langle x_i, Gf \rangle = \sum_{i=1}^k \langle (x \otimes f) G^* x_i, Gf_i \rangle.$$

Let $u_i = G^* x_i \in X^{**}$ and $g_i = Gf_i$, $i = 1, \dots, k$, we get

$$\Phi(T) = \sum_{i=1}^k \langle Tu_i, g_i \rangle$$

for all $T \in \mathcal{F}(X)$, where $Tu_i = T^{**}u_i \in X$.

(2) \Rightarrow (1) From the proof of (1) \Rightarrow (3) we know that a linear functional φ is completely k -rank-nonincreasing if and only if φ_{k+1} is k -rank-nonincreasing. If $\Phi: \mathcal{F}(X) \rightarrow M_N(\mathbb{F})$, then the linear functional $\widehat{\Phi}: M_N(\mathcal{F}(X)) \rightarrow \mathbb{F}$ can be written as

$$\widehat{\Phi}((T_{ij})) = \frac{1}{N} \begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix} \Phi_{N^2}((T_{ij}E_{ij})) \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = A \Phi_{N^2}((T_{ij}E_{ij})) B,$$

where A is a $N^2 \times 1$ matrix with every entry $\frac{1}{N}$ and B is a $1 \times N^2$ matrix with every entry 1. So $(\widehat{\Phi})_n(\cdot) = A^{(n)}\Phi_{nN^2}(\cdot)B^{(n)}$. If $\Phi_{(k+1)N^2}$ is k -rank-nonincreasing, then $(\widehat{\Phi})_{k+1}$ is k -rank-nonincreasing and therefore, $\widehat{\Phi}$, as well as Φ , is completely k -rank-nonincreasing, by Lemma 4.1. \square

Corollary 4.2. *Suppose that X is reflexive, then a bounded linear map $\Phi : \mathcal{F}(X) \rightarrow M_N(\mathbb{F})$ is completely k -rank-nonincreasing if and only if there exist $A_1, \dots, A_r \in \mathcal{B}(X, \mathbb{F}^N)$ and $B_1, \dots, B_r \in \mathcal{B}(\mathbb{F}^N, X)$ with $r \leq k$ such that $\Phi(T) = \sum_{i=1}^r A_i T B_i$, that is, Φ is an elementary operator of length not greater than k .*

Corollary 4.3. *Let $\Phi : \mathcal{F}(X) \rightarrow \mathcal{F}(Y)$ be a bounded linear map. If X is reflexive then Φ is completely k -rank-nonincreasing if and only if Φ is pointwise WOT (or SOT) limit of a net of elementary operators of length not greater than k .*

Proof. Let $A = \{\lambda : \lambda \subset Y \text{ is finite linearly independent set of vectors with norm } 1\}$. For every $\lambda \in A$, take a bounded idempotent operator Q_λ such that its range is the finite dimensional subspace Y_λ spanned by λ . Let $\Phi_\lambda = Q_\lambda \Phi Q_\lambda$. It is clear that $\Phi(T) = \lim_\lambda \Phi_\lambda(T)$ (SOT) for every $T \in \mathcal{F}(X)$. Φ_λ is bounded and completely k -rank-nonincreasing and may be regarded as a map from $\mathcal{F}(X)$ into $\mathcal{B}(Y_\lambda) \simeq M_{N_\lambda}(\mathbb{F})$, where $N_\lambda = \dim(Y_\lambda)$. By Corollary 4.2, there exist $A_{\lambda 1}, \dots, A_{\lambda r} \in \mathcal{B}(X, Y_\lambda)$ and $B_{\lambda 1}, \dots, B_{\lambda r} \in \mathcal{B}(Y_\lambda, X)$ with $r \leq k$ such that $\Phi_\lambda(T) = \sum_{i=1}^r A_{\lambda i} T B_{\lambda i} Q_\lambda$ for $T \in \mathcal{F}(X)$, that is, Φ_λ is an elementary operator of length not greater than k . The converse is obvious. \square

A linear functional φ of $\mathcal{B}(X)$ is said to be σ -weakly continuous if there exist sequences $\{x_i\} \subset X$ and $\{f_i\} \subset X^*$ such that $\sum_i \|x_i\|^2 \|f_i\|^2 < \infty$ and $\Phi(T) = \sum_i \langle T x_i, f_i \rangle$ for all $T \in \mathcal{B}(X)$. The locally convex topology of $\mathcal{B}(X)$ determined by all σ -weakly continuous linear functionals is called the σ -weak operator topology (σ -WOT) of $\mathcal{B}(X)$.

Corollary 4.4. *Let $\Phi : \mathcal{B}(X) \rightarrow \mathcal{B}(Y)$ is a $(\sigma\text{-}w)\text{--}(\sigma\text{-}w)$ continuous linear map. Φ is completely k -rank-nonincreasing if and only if Φ is WOT (or SOT) pointwise limit of a net of elementary operators of length not greater than k .*

Proof. Note that $(\sigma\text{-}w)\text{--}(\sigma\text{-}w)$ continuous linear maps are bounded by closed graph theorem. Define $\Phi_\lambda = Q_\lambda \Phi Q_\lambda$ the same as that in the proof of Corollary 4.3. Regard Φ_λ as $\Phi_\lambda : \mathcal{F}(X) \rightarrow \mathcal{B}(Y_\lambda)$, then Φ_λ has the form in Theorem 4.1. It is easy to see in the proof of Theorem 4.1 that $G^* x_i \in X^{**}$ is continuous with respect to the w^* -topology of X^* if Φ is σ -w continuous, and therefore, we have in fact that $G^* x_i \in X$. It follows that there exist $A_{\lambda 1}, \dots, A_{\lambda r} \in \mathcal{B}(X, Y_\lambda)$ and $B_{\lambda 1}, \dots, B_{\lambda r} \in \mathcal{B}(Y_\lambda, X)$ with $r \leq k$ such that $\Phi_\lambda(T) = \sum_{i=1}^r A_{\lambda i} T B_{\lambda i} Q_\lambda$ for all $T \in \mathcal{F}(X)$, and consequently, holds for all $T \in \mathcal{B}(X)$, since Φ_λ is σ -w continuous and $\mathcal{F}(X)$ is σ -weakly dense in $\mathcal{B}(X)$. \square

Now we are at a position to state and prove the main result in this section which gives a characterization of elementary operators.

Theorem 4.5. *Let X and Y be Banach spaces over real or complex field \mathbb{F} and let $\Phi : \mathcal{B}(X) \rightarrow \mathcal{B}(Y)$ is a linear map. Then Φ is an elementary operator of length not greater than k if and only if Φ is $(\sigma\text{-}w)\text{--}(\sigma\text{-}w)$ continuous and completely k -rank-nonincreasing.*

Proof. “if” part is obvious, we only show the “only if” part.

We may assume that Φ is not completely $(k - 1)$ -rank-nonincreasing. For any $y \in X$ and $g \in Y^*$, $\langle \Phi(T)y, g \rangle$ is $\sigma\text{-}w$ continuous linear functional on $\mathcal{B}(X)$ which is completely k -rank-nonincreasing. By Corollary 4.4, there exist $\xi(y, g) \in X^k$ and $\eta(y, g) \in (X^*)^k$ such that

$$\langle \Phi(T)y, g \rangle = \langle T^{(k)}\xi(y, g), \eta(y, g) \rangle \quad \text{for all } T \in \mathcal{B}(X).$$

Note that $\langle \Phi(T)y, g \rangle$ is a bounded trilinear form. Assume that there exist y_1, y_2 in Y and g_0 in Y^* such that $\eta(y_1, g_0)$ and $\eta(y_2, g_0)$ are linear independent. We may assume that both $\xi(y_1, g_0)$ and $\xi(y_2, g_0)$ are nonzero, otherwise, one can modify $\eta(y_1, g_0)$ and $\eta(y_2, g_0)$ so that they are linearly dependent. Let $y_0 \in Y$ such that $\langle y_0, g_0 \rangle = 1$ and take a finite subspace Y_0 containing y_0, y_1 and y_2 . Let Q be a bounded idempotent with range Y_0 . Then $Q^*g_0 = g_0$. By Corollary 4.4, there exist $W \in \mathcal{B}(X^k, Y)$ and $V \in \mathcal{B}(Y, X^k)$ such that $Q\Phi(T)Q = WT^{(k)}V$. Thus for any $y \in Y_0$ and $g \in Q^*(Y^*)$, we have

$$\langle T^{(k)}\xi(y, g), \eta(y, g) \rangle = \langle \Phi(T)y, g \rangle = \langle Q\Phi(T)Qy, g \rangle = \langle T^{(k)}Vy, W^*g \rangle.$$

So

$$\langle T^{(k)}\xi(y_1, g_0), \eta(y_1, g_0) \rangle = \langle T^{(k)}Vy_1, W^*g_0 \rangle$$

and

$$\langle T^{(k)}\xi(y_2, g_0), \eta(y_2, g_0) \rangle = \langle T^{(k)}Vy_2, W^*g_0 \rangle$$

would hold for every $T \in \mathcal{B}(X)$, this contradicts the assumption that $\eta(y_1, g_0)$ and $\eta(y_2, g_0)$ are linear independent. Similarly, for any y in Y and g_1, g_2 in Y^* , $\xi(y, g_1)$ and $\xi(y, g_2)$ cannot be linearly independent. Therefore, there exists a linear map $\mathbf{B} : Y \rightarrow X^k$ and $\mathbf{C} : Y^* \rightarrow (X^*)^k$ such that

$$\langle \Phi(T)y, g \rangle = \langle T^{(k)}\mathbf{B}y, \mathbf{C}g \rangle$$

holds for all y, g and T . It is clear that \mathbf{B} and \mathbf{C} are bounded and $\mathbf{A} = \mathbf{C}^*|_{X^k} \in \mathcal{B}(X^k, Y)$ since Φ is $(\sigma\text{-}w)\text{--}(\sigma\text{-}w)$ continuous. Hence we get

$$\Phi(T) = \mathbf{A}T^{(k)}\mathbf{B},$$

that is, Φ is an elementary operator of length k . \square

The next corollary is an immediate consequence of Theorem 4.5, Corollary 4.4 and [12, Theorem 2.1].

Corollary 4.6. *Let X and Y be Banach spaces. The following are true:*

(1) *If Φ is an elementary operator of length k , and c is a constant such that $\|\Phi\|_{\text{cb}} < c$, then there are operators $\mathbf{A} \in \mathcal{B}(X^k, Y)$ and $\mathbf{B} \in \mathcal{B}(Y, X^k)$ such that $\|\mathbf{A}\| \|\mathbf{B}\| < c$ and $\Phi(T) = \mathbf{A}T^{(k)}\mathbf{B}$.*

(2) *In the subset of bounded, $(\sigma\text{-}w)$ – $(\sigma\text{-}w)$ continuous linear maps from $\mathcal{B}(X)$ into $\mathcal{B}(Y)$, the set of all elementary operators of length at most k ($k < \infty$) is closed under point-weak limits.*

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